Today: Finish §2.4: Limits and Continuity
+ §2.5: Indeterminate Forms
+ Start §2.7: Limits at Infinity ← will finish in supplementary lecture

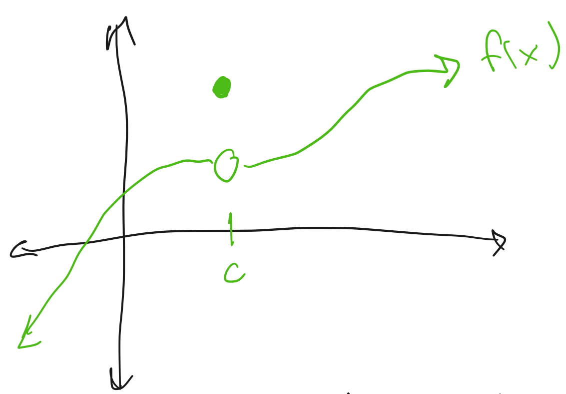
(try it if you have time!)

Warm-up problem Let $f(x) = \begin{cases} x^2 + a & \text{if } x > 3 \\ ax - 2 & \text{if } x \le 3 \end{cases}$

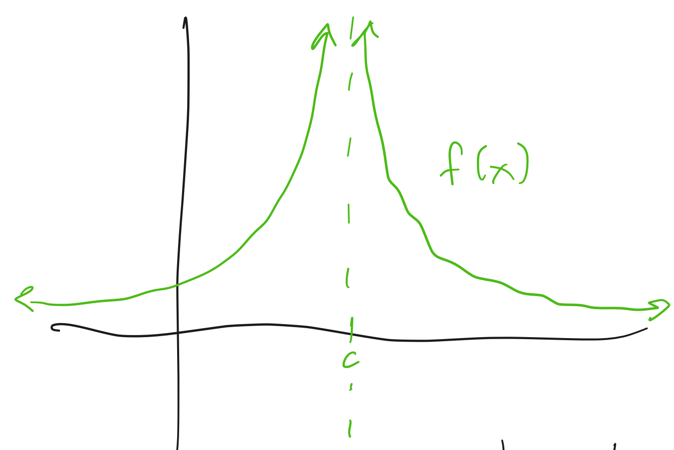
For what value(s) of a is f continuous at $x = 3$?

Last time: A function f is continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

(Geometrically: a discontinuity means there's a hole or a jump in the graph of f)

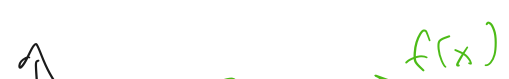


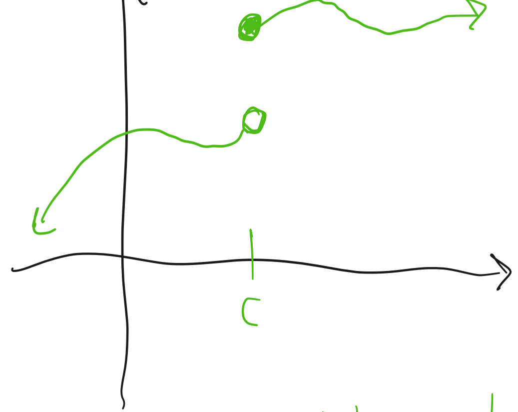
Removeable discontinuity



Infinite discontinuity

Jump discontinuity: if $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exist, but are not equal.





$f(x)$ is right-cont.
$f(x)$ is not left-cont.

Def f is

- **right-continuous** at $x=c$ if $\lim_{x \to c^+} f(x) = f(c)$
- **left-continuous** " " if $\lim_{x \to c^-} f(x) = f(c)$

Warm-up problem Let $f(x) = \begin{cases} x^2 + a & \text{if } x > 3 \\ ax - 2 & \text{if } x \le 3 \end{cases}$

For what value(s) of a is f continuous at $x=3$?

Find: $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x^2 + a = 3^2 + a = 9 + a$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax - 2 = 3a - 2$

$\Rightarrow$ f cont. at $x=3$ if & only if $9 + a = 3a - 2$

$\iff 11 = 2a$

$\iff \frac{11}{2} = a$

Building continuous functions:

Thm (Theorem) (1,4,5) If f and g are continuous at $x=c$, then so are $f+g$, $f-g$, kf, f/g when $g(c) \neq 0$.

k: any constant

Also so are f^{-1} and $f \circ g(x) = f(g(x))$.

(but need to be careful with domain)

Concretely, for $f(g(x))$: if f is continuous at $g(c)$ and g is continuous at c, then $f \circ g$ is continuous at c

Then $f \circ g$ is continuous at c.

Ex $f(x) = 3x \cdot e^{\sin(x^2)} + \cos^2 x$ is continuous at all $x \in \mathbb{R}$ since polynomials, sin, cos, e^x are continuous at all $x \in \mathbb{R}$

§2.5: Indeterminate forms

Q How to evaluate $\lim_{x \to c} f(x)$?

Easy when f is continuous at $x = c$:

$$\lim_{x \to c} f(x) = f(c)$$

Ex
- $\lim_{x \to 2} (x^2 + x) = (2)^2 + (2) = 6$
- $\lim_{x \to \pi/2} \cos(x) = \cos(\pi/2) = 0$

Problem $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ ← $(x-2)(x+2)$

If we substitute $x = 2$, get $\frac{(2)^2 - 4}{(2) - 2} = \frac{0}{0}$ undefined

But can still find the limit:

$$\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2} \right) = \lim_{x \to 2} \left(\underbrace{\frac{x-2}{x-2}}_{f(x)} \underbrace{(x+2)}_{g(x)} \right)$$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x-2}{x-2} = 1$ ($=1$ if $x \neq 2$)

$\lim_{x \to 2} g(x) = (2) + 2 = 4$

$$= \lim_{x \to 2} f(x) \cdot \lim_{x \to 2} g(x) \quad \text{(Product Law)}$$

$$= 1 \cdot 4 = \boxed{4}$$

Def f has an indeterminate form at $x = c$ if $f(c)$ has the form $\frac{0}{0}$, $\infty \cdot 0$, $\infty - \infty$, or $\frac{\infty}{\infty}$

indeterminate

Ex ... $\sin x$ ... 0

Ex $\lim\limits_{x\to 0} \frac{[illegible]}{x} \longrightarrow \frac{0}{0}$

$\lim\limits_{x\to 0} \frac{\cos x}{x} \xrightarrow{\text{not indeterminate}} \frac{1}{0}$ (diverges to ∞)

Ex $\lim\limits_{x\to 9} \frac{x-9}{\sqrt{x}-3} \xrightarrow{\text{indetetermінate}} \frac{9-9}{\sqrt{9}-3} = \frac{0}{0}$

Transform algebraically

$\frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3}$ (multiply by conjugate)

$= \frac{\cancel{(x-9)}(\sqrt{x}+3)}{\cancel{x-9}} = \sqrt{x}+3 \quad \text{if} \quad x \neq 9$

$\lim\limits_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \lim\limits_{x\to 9} \sqrt{x}+3 = \sqrt{9}+3 = \boxed{6}$

Ex $\lim\limits_{x\to 2^+} \left(\cancelto{\infty}{\frac{1}{x-2}} + \cancelto{\infty}{\frac{1}{\sqrt{x}-\sqrt{2}}} \right) = \infty$ (not indeterminate – answer to a student question)

Strategy for indeterminate forms $f(c) = \frac{0}{0}, \infty - \infty, 0 \cdot \infty, \frac{\infty}{\infty}$

- Manipulate algebraically to "cancel the bad part" (and get a continuous function equal to $f(x)$ except at $x = c$)
- Evaluate by substitution

Won't always work: E.g. we'll need other tools for $\lim\limits_{x\to 0} \frac{\sin x}{x}$, which we'll learn in §2.6

Ex $\lim\limits_{x\to 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1} \right)$ $\quad \infty - \infty$

Ex 5 in book

$$\frac{(x+1)}{(x-1)(x+1)} - \frac{2}{(x-1)(x+1)}$$

$$= \frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)}$$

$$\lim_{x \to 1} \frac{1}{x+1} = \frac{1}{(1)+1} = \frac{1}{2}$$